

Elliptic curves and birational representation of Weyl groups

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Abstract

Some Weyl group acts on a family of rational varieties obtained by successive blow-ups at m ($m \geq n+2$) points in the projective space $\mathbb{P}^n(\mathbb{C})$. In this paper we study the case where all the points of blow-ups lie on a certain elliptic curve in \mathbb{P}^n . Investigating the action of Weyl group on the Picard groups on the elliptic curve and on rational varieties, we show that the action on the parameters can be written as a group of linear transformations on the $(m+1)$ -st power of a torus.

1 Introduction

By the works of Coble [1] and Dolgachev-Ortland [2], it has been known that some Weyl group behaves as pseudo-isomorphisms (isomorphisms excluding sub-varieties of co-dimension 2 or higher) and acts on a family of rational varieties obtained by successive blow-ups at m ($m \geq n+2$) points in $\mathbb{P}^n(\mathbb{C})$. Here, the Weyl group is given by the Dynkin diagram in Fig. 1 and denoted by $W(n, m)$.

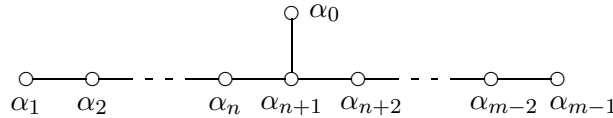


Figure 1: $W(n, m)$ Dynkin diagram

Recently, one of the authors (TT) [7] introduced dynamical systems defined by translations of affine Weyl groups (§6.5 in [3]) with the symmetric Cartan matrices included in $W(n, m)$. For example, if $m \geq n+7$, $W(n, m)$ includes the affine Weyl group of type $E_8^{(1)}$. In the case of $n=2, m=9$, this dynamical system coincides with the elliptic difference Painlevé equation proposed by Sakai [6], from which all the discrete and continuous

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Painlevé equations are obtained by degeneration. For $n \geq 3$, however, the time evolution of the parameters may not be solved in general, thus, the systems should be considered not to be n -dimensional but to be higher dimensional.

On the other hand, Kajiwara *et al.* [4] has proposed a birational representation of the Weyl group $W(n, m)$, in which all the points of blow-ups lie on a certain elliptic curve in \mathbb{P}^n . This representation is a special case of [1, 2]. In this case, the action on the parameters can be written as a group of linear transformations on the $(m + 1)$ -st power of a torus. The calculation was carried out in a rather heuristic manner in [4].

In this paper, we recover the birational representation of Kajiwara *et al.* geometrically, by investigating the actions of the Weyl group on the Picard groups on the elliptic curve and on rational varieties. Our method corresponds to the “linear map” or the “period map” in 2-dimensional case [5, 6].

This article is organized as follows. In Section 2, we review the relationship between rational varieties and groups of Cremona transformations. In Section 3, it is proved that general two elliptic curves in \mathbb{P}^n of degree $n + 1$ are translated to each other by a projective linear transformation. In Section 4, we investigate how the birational representation of the Weyl group is restricted onto elliptic curves of degree $n + 1$ geometrically. In Section 5, we present some examples of calculation, and recover the birational representation of [4].

2 The birational representation of Weyl groups by Coble and Dolgachev-Ortland

Let $m \geq n + 2$. Let $X_{n,m}$ be the configuration space of ordered m points in \mathbb{P}^n :

$$X_{n,m} = \text{PGL}(n+1) \setminus \left\{ \left(\begin{array}{ccc} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{array} \right) \middle| \begin{array}{l} \text{the determinant of} \\ \text{every } (n+1) \times (n+1) \\ \text{sub-matrix is nonzero} \end{array} \right\} / (\mathbb{C}^\times)^m,$$

which is a quasi-projective variety of dimension $n(m - n - 2)$. We also consider $X_{n,m}^1 \simeq X(n, m + 1)$ with a natural projection $\pi : X_{n,m}^1 \rightarrow X_{n,m}$

$$\left(\begin{array}{cccc} a_{01} & \cdots & a_{0m} & x_0 \\ a_{11} & \cdots & a_{1m} & x_1 \\ \vdots & \cdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & x_n \end{array} \right) \mapsto \left(\begin{array}{ccc} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{array} \right),$$

where each fiber is \mathbb{P}^n and $X_{n,m}$ is referred to as the parameter space.

Let $A \in X_{n,m}$ and let X_A be the rational variety obtained by successive blow-ups at the points $P_i = P_i(A) = (a_{0i} : \cdots : a_{ni})$ ($i = 1, 2, \dots, m$) from \mathbb{P}^n . We denote the family of rational projective varieties X_A ($A \in X_{n,m}$) by $\tilde{X}_{n,m}^1$, which also has the natural fibration $\tilde{\pi} : \tilde{X}_{n,m}^1 \rightarrow X_{n,m}$.

Let $E = E(A)$ be the divisor class on X_A of the total transform of a hyper-plane in \mathbb{P}^n and let $E_i = E(A)$ be the exceptional divisor class generated by blow-up at the point

P_i . The group of divisor classes of X_A : $\text{Pic}(X_A) \simeq H^1(X_A, \mathcal{O}^\times) \simeq H^2(X_A, \mathbb{Z})$ (the second equivalence comes from the fact that X_A is a rational projective variety), is described as the lattice

$$\text{Pic}(X_A) = \mathbb{Z}E \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_m. \quad (1)$$

Notice that this cohomology group is independent of A , while X_A is not isomorphic to $X_{A'}$ for $A' (\neq A) \in X_{n,m}$ in general.

Let $e \in H_2(X_A, \mathbb{Z})$ be the class of a generic line in \mathbb{P}^n and let e_i be the class of a generic line in the exceptional divisor of the blow-up at the point P_i . Then, e, e_1, e_2, \dots, e_m consist a basis of $H_2(X_A, \mathbb{Z}) \simeq (H^2(X_A, \mathbb{Z}))^*$ (the Poincaré duality) and the intersection numbers are given by

$$\langle E, e \rangle = 1, \quad \langle E, e_j \rangle = 0, \quad \langle E_i, e \rangle = 0, \quad \langle E_i, e_j \rangle = -\delta_{ij}.$$

Following Dolgachev-Ortland [2], we take the root basis $\{\alpha_0, \dots, \alpha_{m-1}\} \subset H^2(X_A, \mathbb{Z})$ and the co-root basis $\{\alpha_0^\vee, \dots, \alpha_{m-1}^\vee\} \subset H_2(X_A, \mathbb{Z})$ as

$$\begin{aligned} \alpha_0 &= E - E_1 - E_2 - \cdots - E_{n+1}, & \alpha_i &= E_i - E_{i+1} \quad (i > 0) \\ \alpha_0^\vee &= (n-1)e - e_1 - e_2 - \cdots - e_{n+1}, & \alpha_i^\vee &= e_i - e_{i+1} \quad (i > 0), \end{aligned}$$

then, $\langle \alpha_i, \alpha_i^\vee \rangle = -2$ holds for any i and these root bases define the Dynkin diagram of type $T_{2,n+1,m-n-1}$ by assigning a root α_i to every vertex α_i and connecting two distinct vertices α_i and α_j if $\langle \alpha_i, \alpha_j^\vee \rangle = 1$ (in our case $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ or 1 for $i \neq j$) (Fig. 1).

Let us define the root lattice $Q = Q(n, m) \subset H^2(X_A, \mathbb{Z})$ and the co-root lattice $Q^\vee = Q^\vee(n, m) \subset H_2(X_A, \mathbb{Z})$ as $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{m-1}$ and $Q^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_{m-1}^\vee$ respectively. For every α_i the formulae

$$\begin{aligned} r_{\alpha_i*}(D) &= D + \langle D, \alpha_i^\vee \rangle \alpha_i \quad \text{for any } D \in Q \\ r_{\alpha_i*}^\vee(d) &= d + \langle \alpha_i, d \rangle \alpha_i^\vee \quad \text{for any } d \in Q^\vee \end{aligned} \quad (2)$$

define linear involutions (called simple reflections) of the bi-lattice (Q, Q^\vee) and they generate the Weyl group W of type $T_{2,n+1,m-n-1}$, which we denote by $W_*(n, m)$.

These simple reflections correspond to certain birational transformations on the fiber space $\tilde{\pi} : \tilde{X}_{n,m}^1 \rightarrow X_{n,m}$. Let us define birational transformations $r_{i,j}$ ($1 \leq i < j \leq m$) and r_{i_0, i_1, \dots, i_n} ($1 \leq i_0 < \cdots < i_n \leq m$) on the fiber space as:
 $r_{i,j}$ exchanges the points P_i and P_j :

$$r_{i,j} : (\cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{a}_j \mid \cdots \mid \mathbf{x}) \mapsto (\cdots \mid \mathbf{a}_j \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{x}) \quad (3)$$

and r_{i_0, i_1, \dots, i_n} is the standard Cremona transformation with respect to the points $P_{i_0}, P_{i_1}, \dots, P_{i_n}$, i.e. for example, $r_{1,2,\dots,n+1}$ is the composition of a projective transformation and the standard Cremona transformation with respect to the origins $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ as

$$\begin{aligned}
r_{1,2,\dots,n+1} : (A \mid \mathbf{x}) &= (A_{1,\dots,n+1} \mid A_{n+2,\dots,m} \mid \mathbf{x}) \\
&\mapsto A_{1,\dots,n+1}^{-1} (A \mid \mathbf{x}) =: \left(I_{n+1} \mid \cdots \begin{array}{c} \vdots \\ a''_{ij} \\ \vdots \end{array} \cdots \mid \begin{array}{c} \vdots \\ x''_i \\ \vdots \end{array} \right) \\
&\mapsto \left(I_{n+1} \mid \cdots \begin{array}{c} \vdots \\ a''_{ij}^{-1} \\ \vdots \end{array} \cdots \mid \begin{array}{c} \vdots \\ x''_i^{-1} \\ \vdots \end{array} \right),
\end{aligned} \tag{4}$$

where A_{j_1,j_2,\dots,j_k} denotes the $(n+1) \times k$ matrix $(\mathbf{a}_{j_1} \mid \cdots \mid \mathbf{a}_{j_k})$ and I_k denotes the $k \times k$ identity matrix. (In section 5, a''_{ij}^{-1} and x''_i^{-1} are denoted as a'_{ij} and x'_i , respectively.)

Let w denotes the reflection $r_{i,j}$ or r_{i_0,i_1,\dots,i_n} . The reflection w acts on the parameter space $X_{n,m}$ and preserves the fibration $\tilde{\pi} : \tilde{X}_{n,m}^1 \rightarrow X_{n,m}$. Recall that $H^2(X_A, \mathbb{Z})$ is independent of $A \in X_{n,m}$. Hence, w defines an action on this co-homology group. Moreover, the induced birational map $w : X_A \dashrightarrow X_{w(A)}$ for generic $A \in X_{n,m}$ is a pseudo-isomorphism, i.e. an isomorphism except sub-manifolds of co-dimension 2 or higher, and the lines corresponding to the classes e and e_i can be chosen so that they do not meet the excepted part. Since $H_2(X_A, \mathbb{Z})$ is also independent of $A \in X_{n,m}$, w defines an action on this homology group and preserves the intersection form $\langle \cdot, \cdot \rangle : H^2(X_A, \mathbb{Z}) \times H_2(X_A, \mathbb{Z}) \rightarrow \mathbb{Z}$.

The birational maps $r_{i,i+1}$ and $r_{1,2,\dots,n+1}$ correspond to the simple reflections $r_{\alpha_{i*}}$ ($1 \leq i \leq m-1$) and $r_{\alpha_{0*}}$ respectively. Indeed, their push-forward actions $H^2(X_A, \mathbb{Z}) \rightarrow H^2(X_{w(A)}, \mathbb{Z})$ ($w = r_{i,i+1}$ or $r_{1,2,\dots,n+1}$) and $H_2(X_A, \mathbb{Z}) \rightarrow H_2(X_{w(A)}, \mathbb{Z})$ are given by the formulae:

$$\begin{aligned}
r_{i,i+1*}(D) &= D + \langle D, \alpha_i^\vee \rangle \alpha_i \\
r_{i,i+1*}(d) &= d + \langle \alpha_i, d \rangle \alpha_i^\vee \\
r_{1,2,\dots,n+1*}(D) &= D + \langle D, \alpha_0^\vee \rangle \alpha_0 \\
r_{1,2,\dots,n+1*}(d) &= d + \langle \alpha_0, d \rangle \alpha_0^\vee
\end{aligned} \tag{5}$$

for any $D \in H^2(X_A, \mathbb{Z})$ and any $d \in H_2(X_A, \mathbb{Z})$. Recall that root lattice $Q(n, m)$ and co-root lattice $Q^\vee(n, m)$ are subsets of $H^2(X_A, \mathbb{Z})$ and $H_2(X_A, \mathbb{Z})$ respectively and hence the formulae (5) are extensions of (2) onto these (co)-homology groups.

Definition. A hyper-surface in X_A is called nodal if its class is $w_*(\alpha_0)$ for some $w_* \in W_*(n, m)$. Let $N_{n,m}$ denote the set of $A \in X_{n,m}$ such that X_A admits a nodal hyper-surface. We also write $N_{n,m}^1$ and $\tilde{N}_{n,m}^1$ as $\pi^{-1}(N_{n,m})$ and $\tilde{\pi}^{-1}(N_{n,m})$ respectively.

We define $W(n, m)$ as the group generated by $r_{i,i+1}$ ($i = 1, 2, \dots, m-1$) and $r_{1,2,\dots,n+1}$.

Proposition 2.1 (Coble, Dolgachev-Ortland). *Let $m \geq n+2$.*

i) $W(n, m) \simeq W_*(n, m)$ holds. $W(n, m)$ acts on $\tilde{X}_{n,m}^1 \setminus \tilde{N}_{n,m}^1$.

- ii) Each element $w \in W(n, m)$ defines an action on $H^2(X_A, \mathbb{Z})$ and $H_2(X_A, \mathbb{Z})$, and preserves the intersection form $\langle \cdot, \cdot \rangle : H^2(X_A, \mathbb{Z}) \times H_2(X_A, \mathbb{Z}) \rightarrow \mathbb{Z}$.
- iii) The birational maps $r_{i, i+1}$ and $r_{1, 2, \dots, n+1}$ correspond to the simple reflections r_{α_i} ($1 \leq i \leq m-1$) and r_{α_0} respectively. For the reflection $r_\alpha = w \circ r_{\alpha_i} \circ w^{-1}$, where $\alpha = w(\alpha_i)$ is a real root, the formulae

$$\begin{aligned} r_{\alpha*}(D) &= D + \langle D, \alpha^\vee \rangle \alpha \\ r_{\alpha*}(d) &= d + \langle \alpha, d \rangle \alpha^\vee \end{aligned} \tag{6}$$

hold for any $D \in H^2(X_A, \mathbb{Z})$ and any $d \in H_2(X_A, \mathbb{Z})$.

- iv) Every $r_{i, j}$ or r_{i_0, \dots, i_n} is an element of $W(n, m)$.

3 On the embedding of elliptic curve to \mathbb{CP}^n

Let T be an elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and let $\iota : T \rightarrow \mathbb{P}^n$ be an embedding. We define the degree of ι as that of the pull-back of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1) \simeq E$ by ι .

Proposition 3.1. (Algebraic version) *Let ι and ι' be embeddings of T to \mathbb{P}^n of degree $n+1$ s.t. both $\iota(T)$ and $\iota'(T)$ are not contained in any hyper-plane. Then, there exists a translation $\sigma : T \rightarrow T$ and a projective linear transformation $G \in \text{PGL}(n+1)$ s.t. $G \circ \iota = \iota' \circ \sigma$ holds.*

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{G} & \mathbb{P}^n \\ \iota \uparrow & \circlearrowleft & \uparrow \iota' \\ T & \xrightarrow{\sigma} & T \end{array}$$

This proposition can be also stated as follows.

Let $f(u)$ be a holomorphic function on \mathbb{C} . We say that $f(u)$ has the quasi-periodicity if there exist constants l_1, l_τ, c_1, c_τ in \mathbb{C} s.t. the formulae

$$\begin{aligned} f(u+1) &= f(u) \exp\{2\pi\sqrt{-1}(l_1 u + c_1)\} \\ f(u+\tau) &= f(u) \exp\{2\pi\sqrt{-1}(l_\tau u + c_\tau)\} \end{aligned}$$

hold, and we refer such a function to as a theta function for $T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

Proposition 3.2. (Analytic version) *Let ι be a holomorphic map $\iota : \mathbb{C} \rightarrow \mathbb{P}^n : u \mapsto (f_0(u) : f_1(u) : \dots : f_n(u))$ s.t. (i) f_i 's have the same quasi-periodicity, (ii) f_i 's are linearly independent, and (iii) each of them has $n+1$ zero points in the fundamental domain. Let $\iota' : \mathbb{C} \rightarrow \mathbb{P}^n : u \mapsto (f'_0(u) : f'_1(u) : \dots : f'_n(u))$ be also a holomorphic map s.t. f_i 's satisfy (i), (ii), (iii) with the same τ (the quasi-periodicity need not be the same with that of f_i 's). Let u_0, u_1, \dots, u_n be zeros of $f_0(u)$ and let u'_0, u'_1, \dots, u'_n be zeros of $f'_0(u)$. Then, there exists a projective linear transformation $G \in \text{PGL}(n+1)$ s.t. $\iota(u)G = \iota'(u+a)$ holds, where a is determined by*

$$a = \frac{1}{n+1}(u'_0 + u'_1 + \dots + u'_n - u_0 - u_1 - \dots - u_n).$$

Proof. The sum of zeros of $f'_0(u+a)$ is $(u'_0-a) + (u'_1-a) + \cdots + (u'_n-a) = u_0 + u_1 + \cdots + u_n$, then $f_0(u)$ and $f'_0(u+a)$ coincide up to a trivial theta function, i.e. the quasi-periods of $f_0(u)$ and $f'_0(u+a) \exp(\alpha_2 u^2 + \alpha_1 u)$ coincide for some $\alpha_1, \alpha_2 \in \mathbb{C}$. Thus, the quasi-periods of $f_i(u)$'s and $f'_i(u+a) \exp(\alpha_2 u^2 + \alpha_1 u)$'s also coincide. Hence, we can assume f_i 's and f'_i 's have the same quasi-periodicity.

From the Riemann-Roch theorem

$$\dim H^0(T, \mathcal{O}(D)) - \dim H^1(T, \mathcal{O}(D)) = 1 - g + \deg D,$$

where D is a divisor on T , and the Serre duality $H^1(T, \mathcal{O}(D))^* \simeq H^0(T, \Omega(-D))$, we have

$$\dim H^0(T, \mathcal{O}(D)) - \dim H^0(T, \Omega(-D)) = \deg D.$$

Put $D = u_0 + u_1 + \cdots + u_n$ (summation of divisors), then we have $H^0(T, \Omega(-D)) = 0$ and therefore $\dim H^0(T, \mathcal{O}(D)) = n + 1$. Hence, meromorphic functions $f_i(u)/f_0(u)$ ($i = 0, 1, \dots, n$) on T consist a basis of $H^0(T, \mathcal{O}(D))$ and $f'_i(u)/f_0(u)$ ($i = 0, 1, \dots, n$) also consist a basis of $H^0(T, \mathcal{O}(D))$. Hence, there exists a linear transformation G on \mathbb{C}^{n+1} s.t.

$$\left(\frac{f'_0(u)}{f_0(u)}, \dots, \frac{f'_n(u)}{f_0(u)} \right) = \left(\frac{f_0(u)}{f_0(u)}, \dots, \frac{f_n(u)}{f_0(u)} \right) G$$

holds, and therefore,

$$(f'_0(u), \dots, f'_n(u)) = (f_0(u), \dots, f_n(u)) G$$

holds. □

4 Elliptic curves and birational representation of Weyl groups

Let X_A be the rational variety obtained by successive blow-ups of \mathbb{P}^n at points P_i : $\rho_A : X_A \rightarrow \mathbb{P}^n$. Assume that there exist τ , $\text{Im } \tau > 0$, and an embedding $\iota_A : T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \rightarrow \mathbb{P}^n$ s.t. (i) the degree of ι_A is $n+1$; (ii) all P_i are on $\iota_A(T)$ (thus, $\iota_A(T)$ is not contained in any hyper-plane). The embedding ι_A can be lifted to an embedding uniquely $\tilde{\iota}_A : T \rightarrow X_A$ s.t. $\rho_A \circ \tilde{\iota}_A = \iota_A$ holds. We denote by $\tilde{\iota}_A^*$ the pull-back $\tilde{\iota}_A^* : \text{Pic}(X_A) \rightarrow \text{Pic}(T)$.

Proposition 4.1. *For $w \in W(n, m)$, w induces an isomorphism from $\tilde{\iota}_A(T)$ to $w \circ \tilde{\iota}_A(T)$.*

Proof. Since $P_{i_0}, P_{i_1}, \dots, P_{i_n}$ ($1 \leq i_0 < i_1 < \cdots < i_n \leq m$) are not on any hyper-plane, the elliptic curve $\iota_A(T) \subset \mathbb{P}^n$ is not contained in a hyper-plane; therefore the generators $r_{1,2,\dots,n+1}$, $r_{i,i+1}$ act $\iota_A(T)$ birationally. Further, they can be extended to an isomorphism of elliptic curves. Thus, by composition and lifting, the assertion follows. □

Lemma 4.1. *The homology class of $\tilde{\iota}_A(T)$ in $H_2(X_A, \mathbb{Z})$ is $(n+1)e - e_1 - e_2 - \cdots - e_m$.*

Proof. The intersection numbers of $\tilde{\iota}_A(T)$ and the basis of $H^2(X_A, \mathbb{Z})$ are

$$\begin{aligned}\langle E, \tilde{\iota}_A(T) \rangle &= n + 1 \quad (\text{the degree of } \iota_A(T)) \\ \langle E_i, \tilde{\iota}_A(T) \rangle &= 1 \quad (1 \leq i \leq m).\end{aligned}$$

□

Proposition 4.2. *The degree of $w \circ \iota_A : T \rightarrow \mathbb{P}^n$ is $n + 1$.*

Proof. From (5), we have

$$\begin{aligned}\deg(w \circ \iota_A(T)) &= H^2(X_{w(A)}, \mathbb{Z}) \langle E, w \circ \tilde{\iota}_A(T) \rangle_{H^2(X_{w(A)}, \mathbb{Z})} \\ &= \langle E, w_*((n + 1)e - e_1 - e_2 - \cdots - e_m) \rangle \\ &= \langle E, (n + 1)e - e_1 - e_2 - \cdots - e_m \rangle \\ &= n + 1.\end{aligned}$$

□

Notation For $u \in T$, we denote the divisor on T corresponding to u by u again and we denote its class by $[u]$. We denote the addition of divisor classes $[u_1]$ and $[u_2]$ by $[u_1] + [u_2]$.

Let $Y_{n,m}$ denote a subset of $X_{n,m}$:

$$\{A \in X_{n,m} \setminus N_{n,m} ; \exists \tau, \text{Im } \tau > 0, \exists \iota_A : T \rightarrow \mathbb{P}^n : \text{an embedding s.t. } (*)\}$$

where $(*)$ is (i) the degree of ι_A is $n + 1$; (ii) all P_i are on $\iota_A(T)$. From Prop. 4.1 and Prop. 4.2, the action of $W(n, m)$ can be restricted on the fiber space over $Y_{n,m}$. For an embedding of an elliptic curve $\iota_A : T \rightarrow \mathbb{P}^n$ and $w \in W(n, m)$, we write $w \circ \iota_A : T \rightarrow \mathbb{P}^n$ as $\iota_{w(A)}$. It should be noted that ι_A and $\iota_{w(A)}$ are not determined only by A and $w(A)$, respectively, e.g., both $\iota_A = (1, \wp(u), \wp'(u), \dots, \wp^{(n-1)}(u))$ and $\iota'_A = (1, \wp(-u), \wp'(-u), \dots, \wp^{(n-1)}(-u))$ satisfy (i) and (ii).

$$\begin{array}{ccc} X_A & \xrightarrow{\quad w \quad} & X_{w(A)} \\ \uparrow \iota_A & \nearrow \iota_{w(A)} & \\ T & & \end{array}$$

In this section, we consider the action of $W(n, m)$ on $Y_{n,m}$ via the orbit of an embedding of an elliptic curve $\iota_A : T \rightarrow \mathbb{P}^n$ for a parameter $A \in Y_{n,m}$.

We use the following notation (**):

u : a point on T ;

P_i : a point in \mathbb{P}^n determined by the i -th column of the parameter A ;

u_i ($1 \leq i \leq m$): the point on T s.t. $\iota_A(u_i) = P_i$;

E : the total transform of the class of a hyper-plane in X_A ;

v : a point in T s.t. $(n+1)[v] = \tilde{\iota}_A^*(E)$ ($\tilde{\iota}_A^*(E)$ is a line bundle on T of degree $n+1$, and therefore, there exists such a point $v \in T$.);

$A' := w(A)$;

P'_i : a point in \mathbb{P}^n determined by the i -th column of the parameter A' ;

$u' := \iota_{A'}^{-1} \circ w \circ \iota_A(u) = u$.

u'_i ($1 \leq i \leq m$): the point on T s.t. $\iota_A(u'_i) = P'_i$;

E' : the total transform of the class of a hyper-plane in $X_{A'}$;

v' : a point in T s.t. $(n+1)[v'] = \tilde{\iota}_{A'}^*(E')$;

$$\begin{array}{ccc}
 X_A & \xrightarrow{\quad w \quad} & X_{A'} \\
 \iota_A \uparrow & \nearrow \iota_{A'} & \\
 T & & \\
 \text{Pic}(X_A) & \xleftarrow{\quad w^* \quad} & \text{Pic}(X_{A'}) \\
 \downarrow \iota_A^* & \nwarrow \iota_{A'}^* & \\
 \text{Pic}(T) & &
 \end{array}$$

Remark 4.1. In the above diagrams, $P'_i \neq w(P_i)$ may occur, and therefore $u'_i \neq \iota_{A'}^{-1} \circ w \circ \iota_A(u_i)$ also may occur. For example, we have $r_{ij}(P_i) = P_i$ and $P'_i = P_j$ (P'_i is the i -th column of $r_{ij}(A)$).

Theorem 4.1. *Suppose that $w^*(E')$ and $w^*(E'_i)$ are represented as $b_0^0 E + \sum_{j=1}^m b_0^j E_j$ and $b_i^0 E + \sum_{j=1}^m b_i^j E_j$, respectively. The points $u'_i \in T$ ($1 \leq i \leq m$) and $u' \in T$ are given by the formulae:*

$$u'_i = (n+1)b_i^0 v + \sum_{j=1}^m b_i^j u_j \quad (7)$$

$$u' = u. \quad (8)$$

Moreover,

$$(n+1)v' = (n+1)b_0^0 v + \sum_{j=1}^m b_0^j u_j \quad (9)$$

holds.

Proof. From the definition of $\iota_{w(A)}$ (8) is trivial. From the relation $\iota_A^* \circ w^* = \iota_{A'}^* : \text{Pic}(X_{A'}) \rightarrow \text{Pic}(T)$, the divisor class

$$\iota_A^* \circ w^*(E_i) = \iota_A^*(b_i^0 E + \sum_{j=1}^m b_i^j E_j) = (n+1)b_i^0 [v] + \sum_{j=1}^m b_i^j [u_j]$$

coincides with the class

$$\iota_{A'}^*(E_i) = [u'_i],$$

therefore, we have (7) by Abel's theorem. Similarly, since the divisor class

$$\iota_A^* \circ w^*(E) = \iota_A^*(b_0^0 E + \sum_{j=1}^m b_0^j E_j) = (n+1)b_0^0 [v] + \sum_{j=1}^m b_0^j [u_j]$$

coincides with the class

$$\iota_{A'}^*(E) = (n+1)[v'],$$

we have (9). □

5 Representatives

In order to investigate the actions of the Weyl group, we should choose a suitable representative of the parameter A . In this section, we consider realizations of A and ι_A , and study normalizations of $w(A)$ by $\text{PGL}(n+1)$.

Let $A \in Y_{n,m}$, then, there exist τ ($\text{Im } \tau > 0$) and an embedding $\iota_A : T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \rightarrow \mathbb{P}^n$ s.t. (i) the degree of ι_A is $n+1$; (ii) $\iota_A(T)$ contains P_i . We use the notation $(**)$ in the previous section.

Theorem 5.1. *Suppose that $w^*(E')$ is represented as $b_0^0 E + \sum_{j=1}^m b_0^j E_j$. Then, there exists a translation $\sigma : T \rightarrow T : \sigma(u) = u - s$ and a projective linear transformation $G \in \text{PGL}(n+1)$ s.t.*

$$\iota_A \circ \sigma = G \circ \iota_{w(A)} \quad (10)$$

holds, where $s \in T$ satisfies

$$(n+1)s = (n+1)b_0^0 v + \sum_{j=1}^m b_0^j u_j - (n+1)v. \quad (11)$$

Remark 5.1. By equality (10) the following diagram commutes

$$\begin{array}{ccccc} X_A & \xrightarrow{w} & X_{w(A)} & \xrightarrow{G} & X_{G \circ w(A)} \\ \iota_A \uparrow & \nearrow \iota_{w(A)} & & & \uparrow \iota_A \\ T & \xrightarrow{\sigma} & T & & T \end{array}$$

Proof. The relation (10) follows from Prop.3.1. Thus, we have

$$\begin{aligned} (n+1)[v'] &= \iota_{w(A)}^*(E') \\ &= (G^{-1} \circ \iota_A \circ \sigma)^*(E') \\ &= (\iota_A \circ \sigma)^*(E') \\ &= \sigma^*((n+1)[v]) \\ &= (n+1)[v+s]; \end{aligned}$$

therefore, $(n+1)(v' - v) = (n+1)s \in T$ holds. Further, Theorem 4.1 implies the equality (11). □

In order to compute the action of w on a general point in \mathbb{P}^n , we have to determine $G \in \text{PGL}(n+1)$ explicitly. For that purpose, it is sufficient to compute $n+2$ points in \mathbb{P}^n in general position.

i) The points $G(P'_i)$ ($i = 1, 2, \dots, m$) are calculated as

$$\begin{aligned} G(P'_i) &= G \circ \iota_{w(A)}(u'_i) \\ &= \iota_A \circ \sigma(u'_i) \\ &= \iota_A(u'_i - s), \end{aligned}$$

where s and u'_i are given by (11) and (7), respectively.

ii) The point $G \circ \iota_{w(A)}(u)$ ($u \neq u_i$) is calculated as

$$\begin{aligned} G \circ \iota_{w(A)}(u') &= \iota_A \circ \sigma(u') \\ &= \iota_A(u' - s) \\ &= \iota_A(u - s). \end{aligned}$$

The last equality follows from (8).

5.1 Example 1

Let us calculate G for the embedding $\iota_A(u) = {}^t(1, \wp(u), \wp'(u), \dots, \wp^{(n-1)}(u))$. In this case, we can choose v as $v = 0$.

i) For $r_{1,2,\dots,n+1}$ (4). From (10), the action is given by:

$$\begin{aligned} r_{1,2,\dots,n+1}: (A \mid \mathbf{x}) &= (\iota_A(u_1), \iota_A(u_1), \dots, \iota_A(u_m) \mid \mathbf{x}) \\ &\xrightarrow{A_{1,\dots,n+1}^{-1}} A_{1,\dots,n+1}^{-1}(A \mid \mathbf{x}) =: \left(I_{n+1} \mid \cdots \begin{array}{c} \vdots \\ a''_{ij} \\ \vdots \end{array} \cdots \mid \begin{array}{c} \vdots \\ x''_i \\ \vdots \end{array} \right) \\ &\xrightarrow{\text{SCT}} \left(I_{n+1} \mid \cdots \begin{array}{c} \vdots \\ a''_{ij}{}^{-1} \\ \vdots \end{array} \cdots \mid \begin{array}{c} \vdots \\ x''_i{}^{-1} \\ \vdots \end{array} \right) = (A' \mid \mathbf{x}') \\ &\xrightarrow{G} (\bar{A} \mid \bar{\mathbf{x}}) = (\iota_A(u'_1 + s), \iota_A(u'_2 + s), \dots, \iota_A(u'_m + s) \mid \bar{\mathbf{x}}), \end{aligned}$$

where u'_i and s are calculated by Theorem 4.1 and (11) as

$$\begin{aligned} u'_i &= \begin{cases} u_i - \sum_{j=1}^{n+1} u_j & (1 \leq i \leq n+1) \\ u_i & (n+2 \leq i \leq m) \end{cases}, \\ s &= -\frac{n-1}{n+1} \sum_{j=1}^{n+1} u_j. \end{aligned}$$

Thus, $G \in \text{PGL}(n+1)$ is determined by

$$G(I_{n+1}, (\iota_A(0))') = (\iota_A(u'_1 + s), \iota_A(u'_2 + s), \dots, \iota_A(u'_{n+1} + s), \iota_A(s)),$$

where $(\iota_A(0))'$ is

$$(\iota_A(0))' = \left(\left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_2) \cdots \iota_A(u_{n+1}) \\ 0 \\ 1 \end{array} \right|^{-1}, \left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_1) \cdots \iota_A(u_{n+1}) \\ 0 \\ 1 \end{array} \right|^{-1}, \dots, \left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_1) \cdots \iota_A(u_n) \\ 0 \\ 1 \end{array} \right|^{-1} \right).$$

Here, G can be decomposed as $G_2 \circ G_1$, where

$$G_1(I_{n+1}, (\iota_A(0))') = \begin{pmatrix} & 1 \\ I_{n+1} & \vdots \\ & 1 \end{pmatrix}.$$

Thus, G_1 and G_2 are explicitly written as

$$G_1 = \text{diag} \left(\left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_2) \cdots \iota_A(u_{n+1}) \\ 0 \\ 1 \end{array} \right|, \left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_1) \cdots \iota_A(u_{n+1}) \\ 0 \\ 1 \end{array} \right|, \dots, \left| \begin{array}{c} 0 \\ \vdots \\ \iota_A(u_1) \cdots \iota_A(u_n) \\ 0 \\ 1 \end{array} \right| \right)$$

and $G_2^{-1} = (\text{diag}(\overline{A}^{-1} \iota_A(s)))^{-1} \overline{A}^{-1}$; thus, $G_2 = \overline{A} \text{diag}(\overline{A}^{-1} \iota_A(s))$.

ii) For $r_{i,j}$ (3). we have $u'_i = u_j$, $u'_j = u_i$, $u'_k = u_k$ ($k \neq i, j$), $s = 0$, and G is the identity.

5.2 Example 2

Let us calculate G for the embedding proposed by Kajiwara *et al.* [4]:

$$\iota_A(u) = \left(\frac{[u - u_1 - \varepsilon]}{[u - u_1]} : \dots : \frac{[u - u_{n+1} - \varepsilon]}{[u - u_{n+1}]} \right),$$

where $[z]$ is a theta function whose zero points are $\mathbb{Z} + \mathbb{Z}\tau$ with the order 1, and ε is an extra-parameter. It should be noted that $A_{1,\dots,n+1} = (\iota_A(u_1), \dots, \iota_A(u_{n+1})) = I_{n+1}$ holds. In this case, we chose G in a manner that $\iota_{\overline{A}} := G \circ w \circ \iota_A$ is written in the form

$$\iota_{\overline{A}}(u) = \left(\frac{[u - u'_1 - \overline{\varepsilon}]}{[u - u'_1]} : \dots : \frac{[u - u'_{n+1} - \overline{\varepsilon}]}{[u - u'_{n+1}]} \right).$$

From the diagram

$$\begin{array}{ccccc} X_A & \xrightarrow{-w} & X_{w(A)} & \xrightarrow{G} & X_{G \circ w(A)} \\ \iota_A \uparrow & \nearrow \iota_{w(A)} & & & \uparrow \iota_{\overline{A}} \\ T & \xrightarrow{\sigma} & T & & T \end{array}$$

we have

$$\begin{aligned}
(n+1)[v'] &= \iota_{w(A)}^*(E') \\
&= (G^{-1} \circ \iota_{\overline{A}} \circ \sigma)^*(E') \\
&= (\iota_{\overline{A}} \circ \sigma)^*(E') \\
&= \sigma^*((n+1)[\overline{v}]) \\
&= (n+1)[\overline{v} + s],
\end{aligned}$$

and v , v' and \overline{v} are given by

$$\begin{aligned}
(n+1)v &= \varepsilon + \sum_{i=1}^{n+1} u_i \\
(n+1)\overline{v} &= \overline{\varepsilon} + \sum_{i=1}^{n+1} u'_i \\
(n+1)v' &= (n+1)b_0^0 v + \sum_{j=1}^m b_0^j u_j.
\end{aligned}$$

i) For $r_{1,2,\dots,n+1}$. From theorem 4.1, we have

$$u'_i = \begin{cases} u_i + \varepsilon & (1 \leq i \leq n+1) \\ u_i & (n+2 \leq i \leq m) \end{cases},$$

$$(n+1)s = -\varepsilon - \overline{\varepsilon}.$$

Here, we can choose $\overline{\varepsilon}$ as $\overline{\varepsilon} = -\varepsilon$, then we have $s = 0$.

Further, $G \in \text{PGL}(n+1)$ is determined by

$$G \circ w(A_{1,\dots,n+2}) = G(I_{n+1}, (\iota_A(u_{n+2}))') = (I_{n+1}, \iota_{\overline{A}}(u'_{n+2}))$$

and

$$(\iota_A(u_{n+2}))' = \left(\frac{[u_{n+2} - u_1]}{[u_{n+2} - u_1 - \varepsilon]} : \dots : \frac{[u_{n+2} - u_{n+1}]}{[u_{n+2} - u_{n+1} - \varepsilon]} \right).$$

Hence, G is the identity.

ii) For $r_{k,k+1}$, we have

$$u'_i = \begin{cases} u_i & (i \neq k, k+1) \\ u_{k+1} & (i = k) \\ u_k & (i = k+1) \end{cases},$$

and

$$(n+1)s = \begin{cases} \varepsilon - \overline{\varepsilon} & (k \neq n+1) \\ \varepsilon - \overline{\varepsilon} + u_{n+1} - u_{n+2} & (k = n+1) \end{cases}.$$

Here, we can choose $\bar{\varepsilon}$ as

$$\bar{\varepsilon} = \begin{cases} \varepsilon & (k \neq n+1) \\ \varepsilon + u_{n+1} - u_{n+2} & (k = n+1) \end{cases},$$

then we have $s = 0$.

For $r_{k,k+1}$ ($k \neq n+1$), G is the identity. We calculate G for $r_{n+1,n+2}$. From

$$G \begin{pmatrix} 1 & \frac{[u_{n+2}-u_1-\varepsilon]}{[u_{n+2}-u_1]} & 0 \\ & \ddots & \vdots \\ & & 1 & \frac{[u_{n+2}-u_n-\varepsilon]}{[u_{n+2}-u_n]} & 0 \\ & & \frac{[u_{n+2}-u_{n+1}-\varepsilon]}{[u_{n+2}-u_{n+1}]} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{[u_{n+2}-u_1-\varepsilon]}{[u_{n+1}-u_1]} \\ & \ddots & \vdots \\ & & 1 & \frac{[u_{n+2}-u_n-\varepsilon]}{[u_{n+1}-u_n]} \\ & & & 1 & \frac{[-\varepsilon]}{[u_{n+1}-u_{n+2}]} \end{pmatrix},$$

and by decomposing G into $G_1 = G_2 \circ G_2$ as example 1, we have

$$G_1 = \text{diag} \left(-\frac{[u_{n+2}-u_1]}{[u_{n+2}-u_1-\varepsilon]}, \dots, -\frac{[u_{n+2}-u_n]}{[u_{n+2}-u_n-\varepsilon]}, 1 \right) \times \\ \times \begin{pmatrix} 1 & -\frac{[u_{n+2}-u_1-\varepsilon][u_{n+2}-u_{n+1}]}{[u_{n+2}-u_1][u_{n+2}-u_{n+1}-\varepsilon]} \\ & \ddots & \vdots \\ & & 1 & -\frac{[u_{n+2}-u_n-\varepsilon][u_{n+2}-u_{n+1}]}{[u_{n+2}-u_n][u_{n+2}-u_{n+1}-\varepsilon]} \\ & & & 1 & \frac{[u_{n+2}-u_{n+1}]}{[u_{n+2}-u_{n+1}-\varepsilon]} \end{pmatrix}$$

and

$$G_2 = \left(\frac{[u_{n+2}-u_1-\varepsilon]}{[u_{n+1}-u_1]}, \dots, \frac{[u_{n+2}-u_n-\varepsilon]}{[u_{n+1}-u_n]}, \frac{[-\varepsilon]}{[u_{n+1}-u_{n+2}]} \right).$$

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